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MECHANICAL SYSTEMS OSCILLATIONS WITH HYSTERESIS ENERGY DISSIPATION

Ua У роботі, під час розгляду завдань про коливання механічних систем із урахуванням дисипації енергії в матеріалі, передбачається, що у механічній коливальній системі є гістерезисні втрати, які обумовлені пружними недосконалостями матеріалу. У реально існуючої у всіх матеріалах залежності між напруженням та деформацією спостерігається деяке відхилення від лінійного закону. При цьому пружна недосконалість проявляється у вигляді петлі гістерезиса, площа якої характеризує здатність матеріалу поглинати енергію при механічних коливаннях. У статті проаналізовано вплив гістерезисного тертя механічних систем поблизу резонансу. Особливо важливим є узагальнення даного підходу на реальні конструкції з більшим ступенем вільності. У роботі отримані аналітичні залежності, що зв'язують напруження та деформації з урахуванням гістерезисного розсіювання енергії. Для визначення динамічних характеристик механічних систем використовувався підхід із застосуванням варіаційно-сіткових методів побудови функціоналів типу Релея з подальшою їх мінімізацією методами нелінійного програмування.

En In the work, when considering problems of mechanical systems oscillations, taking into account the dissipation of energy in the material, it is assumed that there are hysteresis losses in the mechanical oscillating system, which is caused by elastic imperfections of the material. In the relationship between stress and strain, which actually exists in all materials, there is some deviation from the linear law. In this case, the elastic imperfection manifests itself in the form of a hysteresis loop, the area of which characterises the ability of the material to absorb energy during mechanical vibrations. The article analyses the influence of hysteresis friction of mechanical systems close to resonance. Of particular importance is the generalisation of this approach to real structures with a larger degree of freedom. The work obtains analytical relationships linking stresses and strains taking into account hysteresis energy dissipation. To determine the dynamic properties of mechanical systems, a variational grid approach has been used to construct Rayleigh-type functionals and then minimize them using non-linear programming methods.

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Introduction

Currently, the study of the stress-strain state of mechanical systems for vibration resistance is an urgent task [1 – 3]. The calculation of structures for vibration resistance should be understood as the process of analyzing the stability of a structure against dynamic loads and vibrations that may lead to fatigue failure or reduced performance. In addition, vibration is an important factor that must be taken into account in the design of mechanical systems.

Vibration control helps to improve the reliability, safety and comfort of operation of aircraft and other mechanical structures. The works of the authors [4 - 7] are devoted to the study of vibrations, including those under the influence of aerodynamic loading. Vibrations taking into account the imperfect elasticity of the real material are of great interest in the study of the dynamic behavior of mechanical systems. In [8], the problems of hysteretic energy dissipation in wire ropes are considered.

One of the current areas of development is the use of layered composites, which are increasingly being used in aerospace engineering, automotive engineering and transport engineering [9].

The use of composites provides a high level of strength properties, impact toughness, and high damping properties, due to their ability to dissipate elastic vibration energy. Solving this problem for complex structures in aircraft manufacturing, ship and building construction is impossible without the use of numerical calculation methods [9, 10].

Resonant vibrations taking into account the imperfect elasticity of the real material are of great interest in the study of the dynamic behavior of mechanical systems, the research methodology of which is the subject of this work.

Statement and solution of the problem

The linearized equation of a hysteretic body under simple cyclic deformation has the form [11]:

$$\sigma_{ij} = 3K \cdot (1+\beta_1 I) \cdot \varepsilon \cdot \delta_{ij} + 2G \cdot (1+\delta_1 I) \cdot (\varepsilon_{ij} - \varepsilon \cdot \delta_{ij}) \quad (1)$$

Where the elastic and hysteresis constants are related respectively by the relations:

$$K = \frac{E}{3 \cdot (1-2\mu)}; \qquad G = \frac{E}{2 \cdot (1+\mu)}; \qquad \mu = \frac{3K-2G}{3 \cdot (3K+G)}. \quad (2)$$

$$\alpha_1 = \frac{1}{3} \cdot (\beta_1 \cdot (1-2\mu) + 2 \cdot \delta_1 \cdot (1+\mu)). \quad (3)$$

Where β_1, δ_1 are the coefficient of hysteresis deviations in the dependencies between spherical and deviatoric tensors; α_1 is the coefficient of hysteresis deviation from Hooke's law under cyclic uniaxial tension-compression; K; E; G are the bulk modulus of elasticity, first and second type modulus of elasticity; μ is the Poisson's ratio;

$\varepsilon=\frac{1}{3}\cdot(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})$; $\delta_{i,j}$ are the Kronecker symbols; I is the hysteresis deviation phase shift operator $\frac{1}{2}\pi$; $i,j=1,2,3$.

Let's look at vibrations near resonance. Since single frequency vibrations (principal vibrations) are generated at resonance, the deformation can be considered as harmonic.

$$\varepsilon_{ij}(t)=\varepsilon_{ij}\cdot\cos pt=\varepsilon_{ij}\cdot\cos\psi. \tag{4}$$

In equations (1), the role of the shift operator is formally replaced by the expression

$$I=\operatorname{tg}\psi. \tag{5}$$

If the vibrations are far from resonance, the dynamic problem differs from the static problem only by the presence of inertial forces. The deformation law can be any, so there are no restrictions imposed by the condition (4).

As in the theory of a viscoelastic body [11], we use the correspondence principle, according to which the relationships between constants, as well as the equations, will have the same form as in the theory of elasticity, if by constants we mean operators:

$$\bar{K}=K\cdot(1+\beta_1 I);\qquad \bar{G}=G\cdot(1+\delta_1 I). \tag{6}$$

Then, according to the correspondence principle, equalities (1), (2), (3) will take the form:

$$\sigma_{ij}=3\bar{K}\cdot\varepsilon\delta_{ij}+2\bar{G}\cdot(\varepsilon_{ij}-\varepsilon\delta_{ij}) \tag{7}$$

$$\bar{K}=\frac{\bar{E}}{3\cdot(1-2\bar{\mu})};\quad G=\frac{\bar{E}}{2\cdot(1+\bar{\mu})};\quad \bar{\mu}=\frac{3\bar{K}-2G}{3\cdot(3\bar{K}+G)}. \tag{8}$$

In expanded form, equation (7) can be represented as

$$\sigma_{11}=\frac{2\bar{G}\cdot(1-\bar{\mu})}{1-2\bar{\mu}}\cdot\left[\varepsilon_{11}+\bar{\mu}\frac{\varepsilon_{22}+\varepsilon_{33}}{1-\bar{\mu}}\right],$$
$$\sigma_{22}=\frac{2\bar{G}\cdot(1-\bar{\mu})}{1-2\bar{\mu}}\cdot\left[\varepsilon_{22}+\bar{\mu}\frac{\varepsilon_{33}+\varepsilon_{11}}{1-\bar{\mu}}\right], \tag{9}$$

$$\sigma_{33}=\frac{2\bar{G}\cdot(1-\bar{\mu})}{1-2\bar{\mu}}\cdot\left[\varepsilon_{33}+\bar{\mu}\frac{\varepsilon_{11}+\varepsilon_{22}}{1-\bar{\mu}}\right].$$

Expanding the incoming operators in expression (9) into a Maclaurin series, restricting to the first two terms of the series:

$$\bar{\mu}=\mu+(1+\mu)\cdot(1+2\mu)\cdot\frac{(\beta_1 I-\delta_1 I)}{3};\qquad(10)$$

$$\frac{1}{1-2\bar{\mu}}=\frac{1}{1-2\mu}\left[1+\frac{2\cdot(1+\mu)\cdot(\beta_1 I-\delta_1 I)}{3}\right];\qquad(11)$$

$$\frac{\bar{\mu}}{1-2\bar{\mu}}=\frac{1}{1-2\mu}\left[1+\frac{(1+\mu)\cdot(\beta_1 I-\delta_1 I)}{3\mu}\right].\qquad(12)$$

Substituting the expressions (10) - (12) into the equation (9), taking into account the formula (2) and neglecting the products and squares of small quantities, we obtain an expression for the stress tensor of a body with linear hysteresis:

$$\sigma_{11}=\frac{(1-\mu)\cdot E}{(1+\mu)\cdot(1-2\cdot\mu)}\cdot\left[\begin{array}{l}\varepsilon_{11}+\frac{\mu}{1-\mu}\cdot(\varepsilon_{22}+\varepsilon_{33})+\frac{(1+\mu)\cdot\beta_1\cdot I}{3\cdot(1-\mu)}\cdot(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+\\+\frac{(1-2\cdot\mu)\cdot\delta_1\cdot I}{3\cdot(1-\mu)}\cdot(2\cdot\varepsilon_{11}-\varepsilon_{22}-\varepsilon_{33})\end{array}\right]$$

$$\sigma_{22}=\frac{(1-\mu)\cdot E}{(1+\mu)\cdot(1-2\cdot\mu)}\cdot\left[\begin{array}{l}\varepsilon_{22}+\frac{\mu}{1-\mu}\cdot(\varepsilon_{33}+\varepsilon_{11})+\frac{(1+\mu)\beta_1\cdot I}{3\cdot(1-\mu)}\cdot(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+\\+\frac{(1-2\cdot\mu)\cdot\delta_1\cdot I}{3\cdot(1-\mu)}\cdot(2\varepsilon_{22}+\varepsilon_{33}+\varepsilon_{11})\end{array}\right]\qquad(13)$$

$$\sigma_{33}=\frac{(1-\mu)\cdot E}{(1+\mu)\cdot(1-2\cdot\mu)}\cdot\left[\begin{array}{l}\varepsilon_{33}+\frac{\mu}{1-\mu}\cdot(\varepsilon_{11}+\varepsilon_{22})+\frac{(1+\mu)\cdot\beta_1\cdot I}{3\cdot(1-\mu)}\cdot(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+\\+\frac{(1-2\cdot\mu)\cdot\delta_1\cdot I}{3\cdot(1-\mu)}\cdot(2\varepsilon_{33}-\varepsilon_{11}-\varepsilon_{22})\end{array}\right]$$

$$\sigma_{12}=\frac{E\cdot(1+\delta_1\cdot I)}{1+\mu}\cdot\varepsilon_{12};\qquad\sigma_{23}=\frac{E\cdot(1+\delta_1\cdot I)}{1+\mu}\cdot\varepsilon_{23};$$

$$\sigma_{31}=\frac{E\cdot(1+\delta_1\cdot I)}{1+\mu}\cdot\varepsilon_{31}.$$

Equation (13) can be written in matrix form:

$$\{\sigma\}=[E]\cdot\{\varepsilon\}+\left[E^{*}\right]\cdot I\cdot\{\varepsilon\}.\qquad(14)$$

Here $\{\sigma\},\{\varepsilon\}$ are column vectors of stresses and strains; Where $[E]$ and $[E^*]$ – are square matrices of elastic and hysteresis constants, respectively [3].

If equality $\beta_1 = \delta_1$ is true, we receive:

$$[E^*] = \delta_1 \cdot [E] \tag{15}$$

The stiffness and hysteresis damping matrices for a volumetric finite element can be found using the traditional finite element method (FEM) algorithm as follows:

$$\begin{gathered}[K] = \int_V [D]^T \cdot [E] \cdot [D] dv; \\ [K^*] = \int_V [D]^T \cdot [E^*] \cdot [D] dv.\end{gathered} \tag{16}$$

Where $[D]$ is the dependence matrix between deformations and nodal displacements of a finite element?

The damping matrix will be proportional to the stiffness matrix if condition (15) is satisfied.

$$[K^*] = \delta_1 \cdot [K]. \tag{17}$$

The problem of resonant vibrations of mechanical systems, taking into account the imperfect elasticity of the material, can be solved using expressions (14) - (17).

The main flexibility in the range of power frequencies and modes of vibration lies in the increased vibratory properties of mechanical systems subjected to vibratory pressures. When solving real practical problems, the matrices of masses and rigidities are of large dimensions, which leads to computational difficulties.

This is due to the fact that the study of structures of complex configuration requires a fairly fine discretisation, which increases the order of the matrices.

The method of quasi-static iterations [12] is used for the determination of the first eigenfrequency and shape.

The functionality that needs to be minimised when the quasi-static iteration method is used is as follows:

$$I = \int_V \bar{U} dV - \omega^2 \int_V \bar{K} dV. \tag{18}$$

Where $\bar{U}$ and $\bar{K}$ are the quadratic forms, which are indicated by formulas:

$$\bar{U} = B(c,c) = \frac{1}{2}\sum_{S=1}^{N}\sum_{K=1}^{N} C_{SK} c_K c_S \; ; \tag{19}$$

$$\bar{K} = K(c,c) = \frac{1}{2}\sum_{S=1}^{N}\sum_{K=1}^{N} A_{SK} c_K c_S \; . \tag{20}$$

An iterative process of coordinate descent [12] is used to minimise the functionality (18).Then the functional (19) in the vicinity of k+1 is written in the form:

$$I^{k+1} = (Bc^{k+1}, c^{k+1}) - (K(\omega_1^{(1)})^2 c^k, c^{k+1}) \, . \tag{21}$$

Where

$$(\omega^k)^2 = \frac{B(c^k, c^k)}{K(c^k, c^k)}; \tag{22}$$

$$F_i^k = (\omega^k)^2 (Kc^k, e_i) \, . \tag{23}$$

Then the functional (18) in k+1 nearby is written as:

$$I^{k+1} = (Bc^k + \gamma_i^{k+1} e_i, c^k + \gamma_i^{k+1} e_i) - [\omega^{(k)}]^2 (Kc^k, c^k + \gamma^{k+1} e_i) \, . \tag{24}$$

Or after having transformed:

$$\begin{gathered} I^{k+1} = (Bc^k, c^k) + 2\gamma_i^{k+1}(Bc^k, e_i) + (\gamma_i^{k+1})^2 (Be_i, e_i) - \\ -[\omega^{(k)}]^2 (Kc^k, c^k) - \gamma_i^{k+1}[\omega^{(k)}]^2 (Kc^k, e_i) \end{gathered} \tag{25}$$

The increment of the *k*+1 approximating functional is given by the following equation:

$$\Delta I^{k+1} = 2\gamma_i^{k+1}(Bc^k, e_i) + (\gamma_i^{k+1})^2 (Be_i, e_i) - \gamma_i^{k+1}[\omega_1^{(k)}]^2 (Kc^k, e_i) \, . \tag{26}$$

The step size is determined from the condition of the maximum rate of decrease ΔI^{k+1}

$$\frac{\partial \Delta I^{k+1}}{\partial \gamma_i^{k+1}} = 0; \tag{27}$$

Then

$$2(Bc^k, e_i) + 2\gamma_i^{k+1}(Be_i, e_i) - [\omega_1^{(k)}]^2 (Kc^k, e_i) = 0 \, . \tag{28}$$

Hence, the step size is determined by the ratio

$$\gamma^{k+1} = -\frac{2(Bc^k, e_i) - [\omega_1^{(k)}]^2 (Kc^k, e_i)}{2(Be_i, e_i)} \, . \tag{29}$$

Thus, the iterative process is simplified and the final formula for determining the step has the same structure as for the static problem. To determine the spectrum of eigenfrequencies and modes, it is proposed to use the stiffness increase method, which is based on the use of the minimax properties of the Rayleigh-Ritz functional [9].

When using the stiffness enhancement method for the determination of 2nd and higher eigenfrequencies and eigenmodes, it is necessary to solve the problem of minimisation of the Rayleigh-type functional:

$$I(\vec{\upsilon})_{\vec{\upsilon}\in R^N} = \frac{U_h(\vec{\upsilon}) + c\sum_{n=1}^{l-1}\left(\sum_{i=1}^{N}\frac{\partial K_h}{\partial z_i^{(n)}}\upsilon_i\right)^2}{K_h(\vec{\upsilon})}. \quad (30)$$

The coordinate descent method is also used to minimise the function (30). The stiffness enhancement method is the most efficient and economical method in terms of computational resources compared to the traditional method, where each subsequent shape and frequency is found by minimising on a subspace orthogonal to all previously found eigenvectors. It should be emphasised that the stiffness enhancement method allows the determination of the required number of eigenfrequencies and modes, including multiples, which is important when solving the problem of forced vibrations of mechanical systems.

Conclusions

1. In this paper, the damping matrices for the resonant vibrations of mechanical systems are presented in an analytical form.
2. The equations for the components of the stress tensor are presented, taking into account the imperfect elasticity of the material under simple cyclic deformation.
3. Recommendations are given for the use of the obtained damping matrices in the solution of practical problems using a variety of methods. This will help to avoid the difficulties associated with the creation, storage and use of global matrices of masses and stiffnesses.

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