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L. M. Ryzhkov [1], *D.S., Professor*

VECTOR METHODS FOR DETERMINATION OF RIGID BODY ORIENTATION

Ua Розглянуто векторні методи визначення орієнтації твердого тіла sз використанням інформації про вектори в опорній та пов'язаній із тілом системах координат. В основу аналізу покладено метод найменших квадратів у векторно-матричній формі. Такий підхід дозволяє із єдиних позицій розглянути застосування різних методів розв'язання задачі: матричного (з використанням матриці напрямних косинусів), геометричного (із використанням вектора Гіббса), кватерніонного (із використанням кватерніону повороту). Проведено чисельну оцінку точності восьми алгоритмів визначення орієнтації, отриманих на основі даних методів.

En Vector methods for determining the orientation of a rigid body using information about vectors in the reference and body-related coordinate systems are considered. The analysis is based on the least squares method in vector-matrix form. This approach allows us to consider the application of different methods of solving the problem from a single standpoint: matrix (using the directional cosine matrix), geometric (using the Gibbs vector), quaternion (using the quaternion of rotation). Numerical evaluation of the accuracy of eight orientation determination algorithms obtained on the basis of these methods was performed.

Introduction

Methods for determining orientation by measuring vectors in the reference body-related coordinate systems have found wide application where the use of gyroscopic sensors for a long time is impossible. Here, first of all, it is necessary to note the problem of determining the orientation of satellites. As meters are used sun sensors, Earth sensors, magnetometers, star sensors.

From a mathematical point of view, the least squares method is common to most methods, which allows to take into account the possible measurement errors. The approach proposed by Wahba's problem [1], according to which the matrix of guiding cosines is sought by comparing measured and calculated vectors, proved to be particularly effective. This approach is used in most cases when solving the problem of determining orientation.

Most of the purpose of the analysis is to obtain optimal estimates of the directional cosine matrix [2 - 3] and quaternions [4 - 5], which in this paper are also considered and compared in terms of accuracy of orientation.

[1] *National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"*

Problem statement

The purpose of the analysis is to generalize and compare the algorithms for determining the orientation of a rigid body based on the measurement of vectors.

Problem solution

1. *Matrix method of problem solution*

The problem of determining the orientation of the body is considered as the problem of finding the directional cosine matrix $\boldsymbol{R}$ of the transition from the reference coordinate system to the body-related coordinate system, using information about the projections of vectors in these coordinate systems [6 - 10].

This problem is solved by minimizing the loss function

$$g(\boldsymbol{R}) = \frac{1}{2}\sum_{i=1}^{n}\mu_i \left\|\bar{\boldsymbol{a}}_i - \boldsymbol{R}\bar{\boldsymbol{a}}_{oi}\right\|^2 = \frac{1}{2}\sum_{i=1}^{n}\left\|\boldsymbol{a}_i - \boldsymbol{R}\boldsymbol{a}_{oi}\right\|^2, \quad (1)$$

where $\tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{a}}_{oi}$ – vectors in the body-related and in the reference coordinate systems; $\bar{\boldsymbol{a}}_i = \frac{\tilde{\boldsymbol{a}}_i}{\left\|\tilde{\boldsymbol{a}}_i\right\|}$, $\bar{\boldsymbol{a}}_{oi} = \frac{\tilde{\boldsymbol{a}}_{oi}}{\left\|\tilde{\boldsymbol{a}}_{oi}\right\|}$; $\boldsymbol{a}_i = \sqrt{\mu_i}\,\bar{\boldsymbol{a}}_i$, $\boldsymbol{a}_{oi} = \sqrt{\mu_i}\,\bar{\boldsymbol{a}}_{oi}$; μ_i – weights; n – number of vectors.

The loss function is complemented by a requirement for orthogonality of the matrix $\boldsymbol{R}$ and is written in the form

$$g(\boldsymbol{R}) = \frac{1}{2}\sum_{i=1}^{n} tr\left[\left(\boldsymbol{M}_i - \boldsymbol{R}\boldsymbol{M}_o\right)\left(\boldsymbol{M}_i - \boldsymbol{R}\boldsymbol{M}_o\right)^T\right] + \frac{1}{2}\mathrm{tr}\left[\Lambda(\boldsymbol{R}^T\boldsymbol{R} - \boldsymbol{I})\right], \quad (2)$$

where $\boldsymbol{M}_i = \left[\boldsymbol{a}_1\ \boldsymbol{a}_2 \ldots \boldsymbol{a}_n\right]$; $\boldsymbol{M}_o = \left[\boldsymbol{a}_{o1}\ \boldsymbol{a}_{o2} \ldots \boldsymbol{a}_{on}\right]$; Λ – Lagrange multiplier.

The optimal matrix $\boldsymbol{R}$ that is sought from the condition $\frac{\partial g}{\partial \boldsymbol{R}} = 0$ has the form

$$\boldsymbol{R} = \boldsymbol{Q}\left(\sqrt{\boldsymbol{Q}^T\boldsymbol{Q}}\right)^{-1}, \quad (3)$$

where $\boldsymbol{Q} = \boldsymbol{M}\boldsymbol{M}_o^T$.

We will call this solution “Algorithm №1”. Note that if the matrices are formed on the basis of two vectors in accordance with the TRIAD algorithm, then formula (3) is simplified to the form $\boldsymbol{R} = \boldsymbol{M}\boldsymbol{M}_o^T$.

2. *Geometric method of problem solution*

Specify vectors $\vec{b} = \vec{a} - \vec{a}_o$ that characterize displacements of the ends of the vectors (Fig. 1). This vector is situated in the plane of the noted circle and is perpendicular to the axis of rotation.

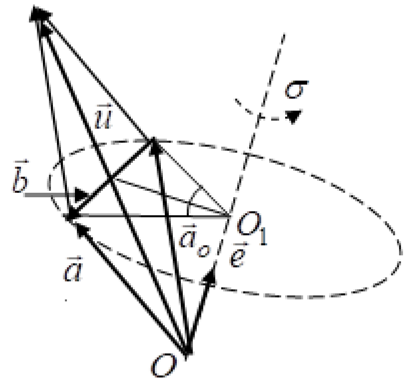

Fig. 1. Vectors

Fig. 1 shows the angle of rotation of the body σ, the axis of rotation of the body and the unit vector $\vec{e}$ of this axis.

Specify the vectors $\vec{u} = \vec{a} + \vec{a}_o$. There is a relation [11]

$$\vec{b} = \vec{g} \times \vec{u} \,. \tag{4}$$

where $\vec{u} = \vec{a} + \vec{a}_o$; $\vec{g} = \vec{e}\,\mathrm{tg}\frac{\sigma}{2}$ – Gibbs vector.

Rewrite the formula (4) in the matrix form

$$\boldsymbol{b} = -\boldsymbol{U}\boldsymbol{g} \quad (i = 1 \div n), \tag{5}$$

where $\boldsymbol{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$.

Determine the vector $\boldsymbol{g}$ minimizing function

$$f = \frac{1}{2}\sum_{i=1}^{n} \left\| \boldsymbol{b}_i + \boldsymbol{U}_i \boldsymbol{g} \right\|^2 . \tag{6}$$

Using dependency $\frac{\partial f}{\partial \boldsymbol{g}} = 0$, we find

$$\boldsymbol{g} = -\boldsymbol{G}^{-1}\boldsymbol{k} \,, \tag{7}$$

where $\boldsymbol{k} = \sum_{i=1}^{n} \boldsymbol{U}_i^T \boldsymbol{b}_i$, $\boldsymbol{G} = \sum_{i=1}^{n} \boldsymbol{U}_i^T \boldsymbol{U}_i$.

Next are the unit rotation vector and the rotation angle of the body (algorithm № 2)

$$\boldsymbol{e} = \frac{\boldsymbol{g}}{\|\boldsymbol{g}\|}; \quad \sigma = 2arctg\left(\|\boldsymbol{g}\|\right). \tag{8}$$

The disadvantage of this algorithm is that it cannot be used for $\sigma = 180^o$.

3. *Quaternion method of problem solution*

3.1. General quaternion solution of the problem

If we put the matrix $\boldsymbol{R}$ in accordance with the quaternion of rotation $\boldsymbol{q}$, we will have the following dependence

$$\boldsymbol{a} = \tilde{\boldsymbol{q}} \circ \boldsymbol{a}_o \circ \boldsymbol{q}, \tag{9}$$

where $\tilde{\boldsymbol{q}}$ – conjugate quaternion.

Rewrite formula (9) as follows

$$\boldsymbol{q} \circ \boldsymbol{a} = \boldsymbol{a}_o \circ \boldsymbol{q} \tag{10}$$

Formula (10) can be written in matrix form [12, 13]

$$\boldsymbol{Vq} = \boldsymbol{V}_o \boldsymbol{q}_o, \tag{11}$$

where

$$\boldsymbol{V} = \begin{bmatrix} 0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & \boldsymbol{D}^T \end{bmatrix}; \; \boldsymbol{V}_o = \begin{bmatrix} 0 & -\boldsymbol{a}_o^T \\ \boldsymbol{a}_o & \boldsymbol{D}_o \end{bmatrix}; \; \boldsymbol{D} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}; \; \boldsymbol{D}_o = \begin{bmatrix} 0 & -a_{oz} & a_{oy} \\ a_{oz} & 0 & -a_{ox} \\ -a_{oy} & a_{ox} & 0 \end{bmatrix}.$$

This expression can be written as follows

$$\boldsymbol{Wq} = \boldsymbol{0}_{4\times1}, \tag{12}$$

where $\boldsymbol{W} = \boldsymbol{V}_o - \boldsymbol{V} = \begin{bmatrix} 0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & \boldsymbol{U} \end{bmatrix} \quad \boldsymbol{U} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$.

For n vectors, you can write the following

$$\boldsymbol{W}_i\boldsymbol{q} = \boldsymbol{0}_{4\times1} \quad \left(i = 1...n\right).$$

Taking into account the errors of measuring vectors, we will solve the problem on the basis of the least squares approach, the essence of which is that we will look for the quaternion $\boldsymbol{q}$ from the condition of minimizing the loss function

$$l\left(\boldsymbol{q}\right) = \frac{1}{2}\sum_{i=1}^{n}\left(\boldsymbol{W}_i\boldsymbol{q}\right)^2 = \frac{1}{2}\boldsymbol{q}^T\boldsymbol{Gq}, \tag{13}$$

where $\boldsymbol{G} = \sum_{i=1}^{n} \boldsymbol{W}_i^T \boldsymbol{W}_i = -\sum_{i=1}^{n} \boldsymbol{W}_i^2 = \sum_{i=1}^{n} \begin{bmatrix} \|\boldsymbol{a}_i\|^2 & \boldsymbol{a}_i^T \boldsymbol{U}_i \\ \boldsymbol{U}_i^T \boldsymbol{a}_i & -\boldsymbol{U}_i^2 \end{bmatrix}$.

Consider the condition that the quaternion $\boldsymbol{q}$ must be normalized and take the following loss function $l(\boldsymbol{q})$:

$$l(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^T \boldsymbol{G}\boldsymbol{q} - \lambda\left(\boldsymbol{q}^T \boldsymbol{q} - 1\right), \tag{14}$$

where λ – Lagrange multiplier.

From the condition $\frac{\partial l(\boldsymbol{q})}{\partial \boldsymbol{q}} = \boldsymbol{G}\boldsymbol{q} - \lambda\boldsymbol{q} = \boldsymbol{0}_{4\times1}$ we find

$$\boldsymbol{G}\boldsymbol{q} = \lambda\boldsymbol{q}. \tag{15}$$

Then you can write

$$l(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^T \boldsymbol{G}\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}^T \lambda\boldsymbol{q} = \frac{1}{2}\lambda. \tag{16}$$

This means that we are interested in the minimum value of the parameter λ. Thus, the problem of finding a quaternion $\boldsymbol{q}$ is equivalent to finding the eigenvector (quaternion) of the matrix $\boldsymbol{G}$, which corresponds to the minimum value of the eigenvalue λ (algorithm №3). In the Matlab environment, it is convenient to use the "*eig*" function .

3.2. Using the Gibbs vector

Let's look at other options for solving the problem in which this function is not used.

Writing the quaternion as a vector $\boldsymbol{q} = [q_0 \; \boldsymbol{q}_v^T]$, expression (15) can be written as

$$\begin{bmatrix} b & \boldsymbol{Z}^T \\ \boldsymbol{Z} & \boldsymbol{H} \end{bmatrix} \begin{bmatrix} q_0 \\ \boldsymbol{q}_v \end{bmatrix} = \lambda \begin{bmatrix} q_0 \\ \boldsymbol{q}_v \end{bmatrix}, \tag{17}$$

where $\boldsymbol{H} = -\sum_{i=1}^{n} \boldsymbol{U}_i^2$; $\boldsymbol{Z} = \sum_{i=1}^{n} \boldsymbol{U}_i^T \tilde{\boldsymbol{a}}_i$; $b = \sum_{i=1}^{n} \|\boldsymbol{a}_i\|^2$.

The system (17) can be written as follows

$$\begin{aligned} q_0 b + \boldsymbol{Z}^T \boldsymbol{q}_v &= \lambda q_0; \\ q_0 \boldsymbol{Z} + \boldsymbol{H}\boldsymbol{q}_v &= \lambda \boldsymbol{q}_v. \end{aligned} \tag{18}$$

Similarly to [5], consider the second equation of this system. Let's rewrite it in the form

$$(\lambda\boldsymbol{I} - \boldsymbol{H})\boldsymbol{Y} = \boldsymbol{Z},$$

where $\boldsymbol{Y} = \dfrac{\boldsymbol{q}_v}{q_0}$ – Gibbs vector.

That is,

$$\boldsymbol{Y} = \left(\lambda \boldsymbol{I} - \boldsymbol{H}\right)^{-1} \boldsymbol{Z} . \tag{19}$$

If you put $\lambda = \lambda_{\min}$, you can find the vector $\boldsymbol{Y}$, and then find the quaternion $\boldsymbol{q}$ (algorithm №4)

$$\boldsymbol{q} = \frac{1}{\sqrt{1+\left|\boldsymbol{Y}\right|^2}} \begin{bmatrix} 1 \\ \boldsymbol{Y} \end{bmatrix}. \tag{20}$$

3.3. Solutions of the problem in quaternions

The eigenvalues of the matrix $\boldsymbol{G}$ are the roots of the characteristic equation

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0, \tag{21}$$

where [14]

$c_1 = -T_1;\ c_2 = -(c_1T_1 + T_2);\ c_3 = -(c_2T_1 + c_1T_2 + T_3);$

$c_4 = \det\boldsymbol{G};\ T_1 = tr\boldsymbol{G};\ T_2 = tr\boldsymbol{G}^2;\ T_3 = tr\boldsymbol{G}^3.$

A similar equation obtain for the matrix $\boldsymbol{H}_{3\times 3}$

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{22}$$

where in the expressions for the coefficients the matrix $\boldsymbol{G}$ must be replaced by a matrix $\boldsymbol{H}$.

According to the Hamilton-Kelly theorem, each matrix corresponds to its characteristic equation, i.e. we can write

$$\boldsymbol{H}^3 + c_1\boldsymbol{H}^2 + c_2\boldsymbol{H} + c_3 = 0. \tag{23}$$

Let us represent the expression $\left(\boldsymbol{H} - \lambda\boldsymbol{I}\right)^{-1}$ as follows

$$\left(\boldsymbol{H} - \lambda\boldsymbol{I}\right)^{-1} = \gamma^{-1}\left(\alpha + \beta\boldsymbol{H} + \boldsymbol{H}^2\right). \tag{24}$$

Here we find

$$\boldsymbol{H}^3 + \left(\beta - \lambda\right)\boldsymbol{H}^2 + \left(\alpha - \beta\lambda\right)\boldsymbol{H} - \left(\gamma + \alpha\lambda\right) = 0. \tag{25}$$

Equating in expressions (23) and (25) the coefficients at the same powers of $\boldsymbol{H}$, we obtain

$$\beta = c_1 + \lambda;\ \alpha = c_2 + \beta\lambda;\ \gamma = -(c_3 + \alpha\lambda). \tag{26}$$

Then you can write

$$\boldsymbol{Y} = \boldsymbol{L} / \gamma, \tag{27}$$

where $\boldsymbol{L} = -\left(\alpha \boldsymbol{I} + \beta \boldsymbol{H} + \boldsymbol{H}^2\right)\boldsymbol{Z}$.

Thus

$$\boldsymbol{q} = \frac{1}{\sqrt{\gamma^2 + |\boldsymbol{L}|^2}} \begin{bmatrix} \gamma \\ \boldsymbol{L} \end{bmatrix}. \tag{28}$$

It is significant that in expression (28) the Gibbs vector is absent (algorithm №5). This algorithm is similar to the QUEST algorithm [5].

3.4. Simplified solutions of the problem in quaternions

Given that $\lambda_{\min} \approx 0$, the minimal eigenvalue can be found from the simplified characteristic equation λ (algorithm № 6)

$\Delta \approx c_3 \lambda_{\min} + c_4 = 0$.

Then

$$\lambda_{\min} \approx -\frac{c_4}{c_3}. \tag{29}$$

Taking into account that $\lambda_{\min} \approx 0,$ you can reduce the amount of calculations by taking into account the calculations $\lambda_{\min} = 0$, That is, for practical calculations can be taken $\lambda_{\min} = 0$. In this case

$$\alpha \boldsymbol{I} + \beta \boldsymbol{H} + \boldsymbol{H}^2 = \gamma \boldsymbol{H}^{-1}$$

that is, $\boldsymbol{L} = -\gamma \boldsymbol{H}^{-1} \boldsymbol{Z}.$

In this case (algorithm № 7),

$$\boldsymbol{q} = \frac{1}{\sqrt{1 + |\boldsymbol{X}|^2}} \begin{bmatrix} 1 \\ \boldsymbol{X} \end{bmatrix}. \tag{30}$$

where $\boldsymbol{X} = -\boldsymbol{H}^{-1}\boldsymbol{Z} = \left(\sum_{i=1}^{n} \boldsymbol{U}_i^T \boldsymbol{U}_i \right)^{-1} \sum_{i=1}^{n} \boldsymbol{U}_i^T \boldsymbol{a}_i$.

Let us numerically estimate the accuracy of the algorithms for various orientation angles and vector values in the reference coordinate system. Let us assume $n = 2$ and write the measured vectors in the form $\boldsymbol{a}_1 = \boldsymbol{E}_1 \boldsymbol{R} \boldsymbol{a}_{o1},\ \boldsymbol{a}_2 = \boldsymbol{E}_2 \boldsymbol{R} \boldsymbol{a}_{o2}$, where the matrices

$$E_1 = \begin{bmatrix} 0,99 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1,01 \end{bmatrix}; \quad E_2 = \begin{bmatrix} 1,01 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0,99 \end{bmatrix}$$

set the measurement errors of the vectors. The weight coefficients are assumed to be unit.

Vector measurement errors (in degrees) are given in Table 1.

***Table* 1.**

Vector measurement errors

№ algor	$\psi = 45^0, \theta = 20^{045}, \varphi = -10^0$ $\tilde{\boldsymbol{a}}_{01} = [1\ 0\ 10]^T$ $\tilde{\boldsymbol{a}}_{02} = [9\ 1\ 0]^T$	$\psi = -10^0, \psi = 30^0, \psi = 10^0$ $\tilde{\boldsymbol{a}}_{01} = [1\ -2\ 1]^T$ $\tilde{\boldsymbol{a}}_{02} = [4\ 1\ 2]^T$	$\psi = 10^0, \psi = 30^0, \psi = -10^0$ $\tilde{\boldsymbol{a}}_{01} = [1\ -2\ -1]^T$ $\tilde{\boldsymbol{a}}_{02} = [-4\ 1\ 2]^T$
1	$\Delta_\psi = -0,17549$ $\Delta_\theta = -0,27951$ $\Delta_\varphi = 0,11605$	$\Delta_\psi = 0,57767$ $\Delta_\theta = -0,22937$ $\Delta_\varphi = 0,70444$	$\Delta_\psi = 0,28006$ $\Delta_\theta = -0,10123$ $\Delta_\varphi = -0,057046$
2	$\Delta_\psi = -0,18605$ $\Delta_\theta = -0,30568$ $\Delta_\varphi = 0,092515$	$\Delta_\psi = 0,5621$ $\Delta_\theta = -0,23186$ $\Delta_\varphi = 0,71836$	$\Delta_\psi = 0,030302$ $\Delta_\theta = -0,027703$ $\Delta_\varphi = -0,14501$
3	$\Delta_\psi = -0,18604$ $\Delta_\theta = -0,30568$ $\Delta_\varphi = 0,092514$	$\Delta_\psi = 0,5621$ $\Delta_\theta = -0,23186$ $\Delta_\varphi = 0,71836$	$\Delta_\psi = 0,030435$ $\Delta_\theta = -0,027044$ $\Delta_\varphi = -0,14556$
4	$\Delta_\psi = -0,18604$ $\Delta_\theta = -0,30568$ $\Delta_\varphi = 0,092514$	$\Delta_\psi = 0,5621$ $\Delta_\theta = -0,23186$ $\Delta_\varphi = 0,71836$	$\Delta_\psi = 0,030435$ $\Delta_\theta = -0,027044$ $\Delta_\varphi = -0,14556$
5	$\Delta_\psi = -0,18604$ $\Delta_\theta = -0,30568$ $\Delta_\varphi = 0,092514$	$\Delta_\psi = 0,5621$ $\Delta_\theta = -0,23186$ $\Delta_\varphi = 0,71836$	$\Delta_\psi = 0,030435$ $\Delta_\theta = -0,027044$ $\Delta_\varphi = -0,14556$
6	$\Delta_\psi = -0,18604$ $\Delta_\theta = -0,30568$ $\Delta_\varphi = 0,092514$	$\Delta_\psi = 0,5621$ $\Delta_\theta = -0,23186$ $\Delta_\varphi = 0,71836$	$\Delta_\psi = 0,030447$ $\Delta_\theta = -0,026983$ $\Delta_\varphi = -0,14561$
7	$\Delta_\psi = -0,18605$ $\Delta_\theta = -0,30568$ $\Delta_\varphi = 0,092515$	$\Delta_\psi = 0,5621$ $\Delta_\theta = -0,23186$ $\Delta_\varphi = 0,71836$	$\Delta_\psi = 0,030302$ $\Delta_\theta = -0,027703$ $\Delta_\varphi = -0,14501$

№ algor	$\psi = 45^0, \theta = 20^{045}, \varphi = -10^0$ $\tilde{a}_{01} = [1\ 0\ 10]^T$ $\tilde{a}_{02} = [9\ 1\ 0]^T$	$\psi = -10^0, \psi = 30^0, \psi = 10^0$ $\tilde{a}_{01} = [1\ -2\ 1]^T$ $\tilde{a}_{02} = [4\ 1\ 2]^T$	$\psi = 10^0, \psi = 30^0, \psi = -10^0$ $\tilde{a}_{01} = [1\ -2\ -1]^T$ $\tilde{a}_{02} = [-4\ 1\ 2]^T$
QUEST	$\Delta_\psi = 0{,}099618$ $\Delta_\theta = -0{,}55806$ $\Delta_\varphi = 0{,}043499$	$\Delta_\psi = -0{,}06602$ $\Delta_\theta = -0{,}034716$ $\Delta_\varphi = 0{,}2166$	$\Delta_\psi = 0{,}053986$ $\Delta_\theta = -0{,}57994$ $\Delta_\varphi = 0{,}18177$

Conclusions

It follows from the calculations that the accuracy of the above algorithms is very close to each other. At the same time, the accuracy of the QUEST algorithm is somewhat different from the accuracy of these algorithms, For practical calculations in quaternion algorithms can be taken $\lambda_{\min} = 0,$ that allows you to use simplified algorithms. The most common quaternion algorithms is an algorithm that is based on finding the eigenvalues and eigenvectors of the matrix $\boldsymbol{G}$.

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