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ON THE USE OF NONLINEARITY CHARACTERISTICS OF THE PENDULUM OSCILLATORY SYSTEM TO DETERMINE THE INITIAL CONDITIONS OF ITS MOVEMENT

Ua   Розглядається поведінка маятникової коливальної системи під час дії навколо осі її обертання постійного неконтрольованого моменту. Показано, що визначити порізно початкове відхилення коливальної системи і зміщення центру її коливань під дією моменту протягом одного виміру - можна, аналізуючи нелінійність спостережуваних коливань. Промодельовані алгоритми, засновані на розкладанні гармоніки у ряд із утриманням двох членів розкладання. Зроблено висновки про межі запропонованої методики.

Ru   Рассматривается поведение маятниковой колебательной системы при действии вокруг оси ее вращения постоянного неконтролируемого момента. Показано, что определить порознь начальное отклонение колебательной системы и смещение центра ее колебаний под действием момента в течение одного измерения – можно, анализируя нелинейность наблюдаемых колебаний. Промоделированы алгоритмы, основанные на разложении гармоники в ряд с удержанием двух членов разложения. Сделаны выводы о границах предлагаемой методики.

Introduction

To reduce the time for determining the initial conditions of motion of oscillatory systems (OS), the method proposed in [1] is widely used. It consists in identifying the initial conditions according to the results of monitoring the motion of the oscillatory system. The presence of a constant uncontrolled impact shifts the center of oscillation of the system, introducing an error in the result. In the monograph [2] considered several ways to deal with this error. This is a

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measurement with two different known oscillation frequencies and a subsequent algorithmic account of the correction, or the finding of stable and unstable equilibrium positions (for pendulum systems) with the subsequent construction of the normal to the bisector of the angle between these two equilibrium positions. The considered methods suggest the presence of two measurement cycles: either with parameters providing different values of the natural frequency, or with initial conditions that differ by 1800. In both cases, this at least doubles the measurement time. In [6], in order to reduce the measurement time, it is proposed to use the acceleration mode of the rotor of the pendulum gyroscopic oscillatory system.

Formulation of the problem

This article explores the possibility of determining the position of the center of oscillations of the pendulum system during one measurement effect cycle by analyzing the nonlinearity that occurs in the case of a constant uncontrolled [3].

Main part

The equation of motion of the pendulum oscillatory system has the form

$$\ddot{\alpha} + \omega^2 \cdot \sin(\alpha) = m, \tag{1}$$

where α is the angle of deviation of the pendulum from the position of stable equilibrium (from the position of the true vertical), $\omega^2 = \mu/l$ is the square of the natural frequency of oscillations of the pendulum, μ is the pendulum, I is the axial moment of inertia of the pendulum, $m = M/I$, M is a constant uncontrolled moment around the axis of suspension of the pendulum.

In the case of "small" angles α, the expansion is $\sin(\alpha) \approx \alpha$, then the solution of equation (1) under the initial conditions

$$t = 0, \quad \alpha = \alpha_0, \quad \dot{\alpha} = \dot{\alpha}_0 \tag{2}$$

has the appearance

$$\alpha = \dot{\alpha}_0 \cdot \omega^{-1} \cdot \sin(\omega \cdot t) + (\alpha_0 - m \cdot \omega^{-2}) \cdot \cos(\omega \cdot t) + m \cdot \omega^{-2}. \tag{3}$$

Since the initial condition α_0 – is unknown, and the "observable" coordinate is $\alpha - \alpha_0$, expression (3) is represented as follows:

$$\alpha - \alpha_0 = \dot{\alpha}_0 \cdot \omega^{-1} \cdot \sin(\omega \cdot t) + (\alpha_0 - m \cdot \omega^{-2})(\cos(\omega \cdot t) - 1). \tag{4}$$

It can be seen from (4) that by observing the motion of the pendulum within the linear model corresponding to (1), one can find an estimate of the angle $\alpha_0 - m\omega^2$ between the initial position of the pendulum and the vertical posi-

tion shifted under the action of a constant uncontrollable moment around the axis of suspension, and not true value of α_0.

Let us try to find α_0 and $m\omega^{-2}$ separately, based on the assumption that the presence of a constant uncontrollable moment distorts the harmonic nature of the motion of the pendulum and the degree of this distortion depends on the magnitude of the moment and the amplitude of oscillations. To confirm this, we present graphs of the dependences of the function obtained as a result of integrating equation (1) at three different positions of stable equilibrium and the same initial deviations of the pendulum from these equilibria (fig. 1), as well as at three different values of the initial deviation of the pendulum positions of stable equilibrium (fig. 2).
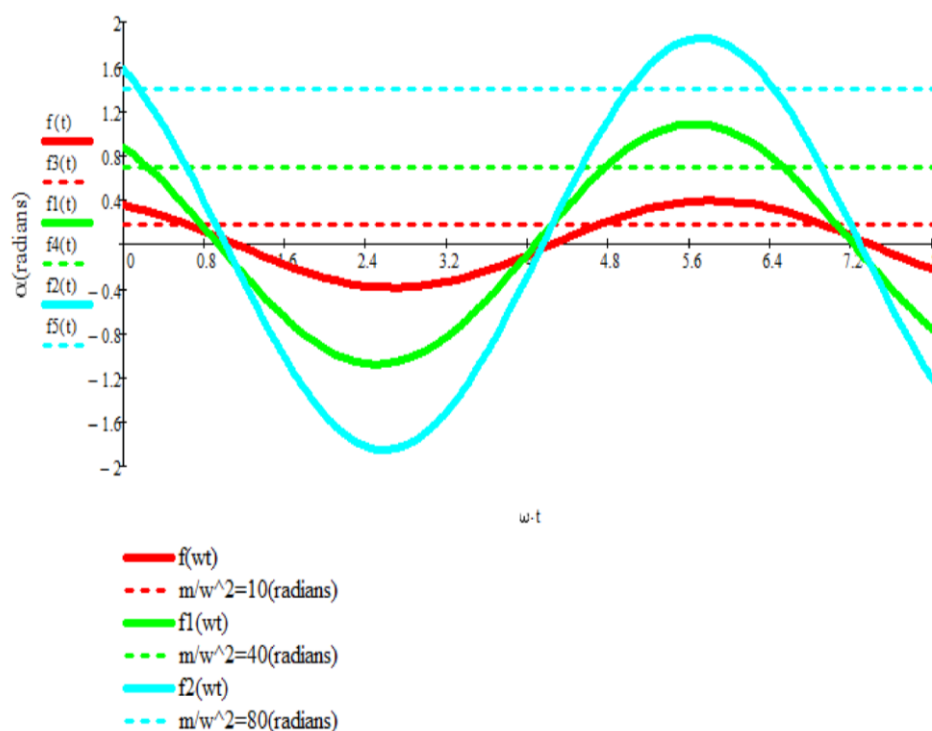

Fig. 1. The dependence of the nature of the oscillations of the pendulum from the position of stable equilibrium

The analysis of fig. 1 and fig. 2 allows us to conclude that the degree of distortion of the harmonic nature of the motion of the pendulum and the period of free oscillations increases significantly with increasing magnitudes of the moment and amplitude of oscillations.

Using the perturbation method, we obtain a solution for the observed coordinate $(\alpha - \alpha_0)$ in the form:
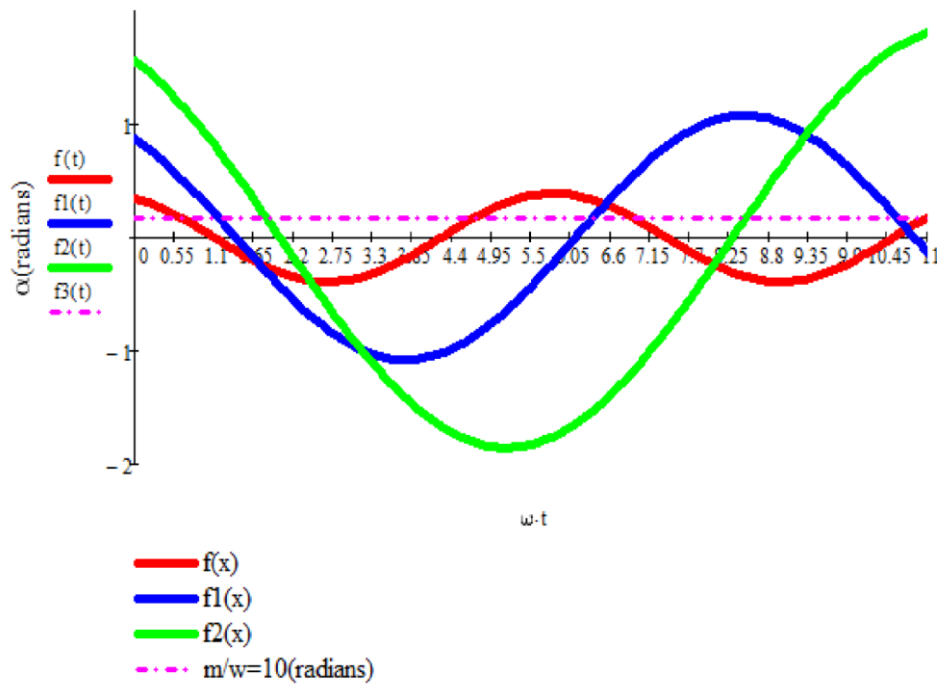
Fig. 2. The dependence of the nature of the oscillations of a pendulum
from the amplitude

$$\alpha - \alpha_0 = (\frac{m}{\omega^{-2}} - \alpha_0) + a \cdot \cos(\psi_0) \cdot \cos(\omega \cdot t) - a \cdot \sin(\psi_0) \cdot \sin(\omega \cdot t) -$$

$$- \frac{a^2}{12} \cdot \frac{m}{\omega^2} \cdot \cos(2\omega \cdot t) \tag{5}$$

Considering notation for unknowns

$$R_1 = \frac{m}{\omega^2} - \alpha_0; \qquad R_2 = a \cdot \cos(\psi_0);$$

$$R_3 = -a \cdot \sin(\psi_0); \qquad R_4 = -\frac{a^2}{12} \cdot \frac{m}{\omega^2} \tag{6}$$

and known functions of time

$$f(t) = \alpha - \alpha_0; \qquad f_1(t) = 1; \qquad f_2(t) = \cos(\omega \cdot t);$$

$$f_3(t) = \sin(\omega \cdot t); \qquad f_4(t) = \cos(2 \cdot \omega \cdot t) \tag{7}$$

expression (5) takes the form:

$$f(t) = R_1 \cdot f_1(t) + R_2 \cdot f_2(t) + R_3 \cdot f_3(t) + R_4 \cdot f_4(t). \tag{8}$$

Consider the investigated oscillatory system as a system with four inputs
$R_1 \div R_4$ and one output x_i. The output value x_i is measured at "n" discrete

points in time, and for these moments the coefficients $f_{1i} \div f_{4i}$ are known. Assuming that the number of measurements of the output value exceeds the number of input values ($n > 4$), we write the matrix form of the overdetermined system of n linear equations with 4 unknowns:

$$X = A \cdot R, \tag{9}$$

where X, R are respectively n and 4 vectors, $A - (n \cdot 4)$ matrix.

According to the least squares method, the best estimate $\hat{R}$ of the unknown vector R is obtained by performing the operations with the available information in accordance with the algorithm:

$$\hat{R} = (A^T \cdot A) \cdot A^T \cdot X \tag{10}$$

Using expression (6), it is not difficult by the found estimates $\hat{R}_1 \div \hat{R}_4$ to find the best estimates for true unknowns.

$$\hat{a} = \sqrt{\hat{R}_2 + \hat{R}_3}; \quad \hat{\psi}_0 = \text{arctg}(-\frac{\hat{R}_3}{\hat{R}_2});$$

$$\frac{\hat{m}}{\omega^2} = -\frac{12 \cdot \hat{R}_4}{\hat{R}_2^2 + \hat{R}_3^2}; \quad \hat{\alpha}_0 = -\frac{12 \cdot \hat{R}_4}{\hat{R}_2^2 + \hat{R}_3^2} - \hat{R}_1. \tag{11}$$

Produce a machine study of the health and basic properties of the proposed algorithms. The oscillatory system will be simulated by solving its differential equation (1), noisy noise in the form of "white noise" intensity δ_m.

The impact of recruitment time

Fig. 3 shows the dependence of the error $\hat{\alpha}_0$ on the time of information collection. A computer experiment was conducted with a OS, the oscillation period of which is $T = 540 \, \text{c}$. For three different combinations of $\frac{m}{\omega^2}$ and α_0, the work of the algorithms was simulated depending on the time of information gathering. From fig. 3 it can be seen that at small intervals of observation of information $T_{\text{н}} \leq (0,1 \div 0,2) \cdot T$, the estimate error $\Delta\hat{\alpha}_0$ is quite large and has a sharply falling character, which passes into an oscillatory one with increasing observation interval.

The influence of the initial deviation of the OS relative to the equilibrium position

Machine modeling of the algorithms allows us to reveal another interesting feature of the investigated algorithm - the minimum error of the determina-

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tion of the vertical at the initial angles of deviation of the CS from the center of the vibrational order of 10 °. This fact may have the following explanation. The
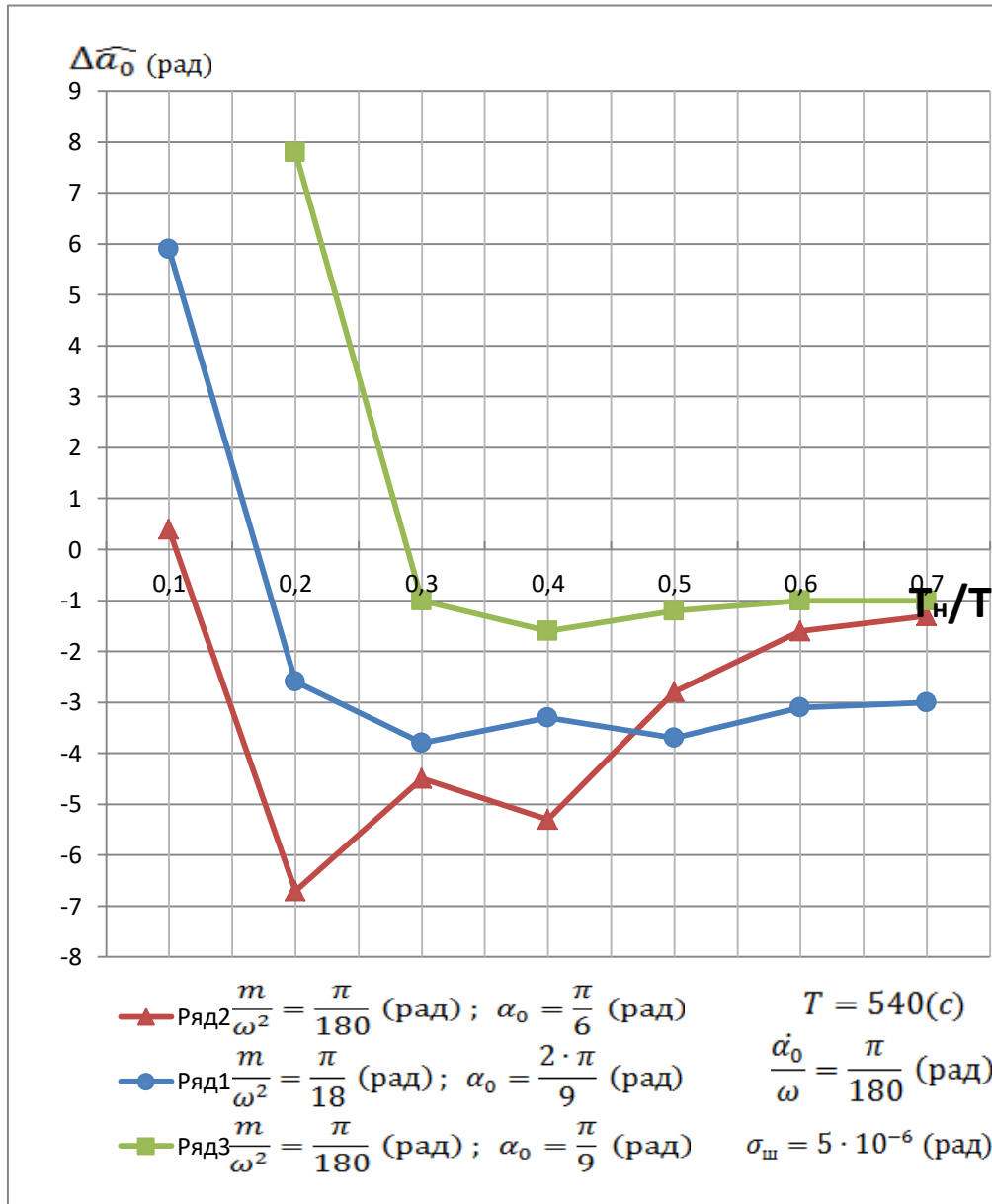

Fig. 3. Dependence of error $\Delta\hat{\alpha}_0$ estimates $\hat{\alpha}_0$ of the initial deviation of the OS from the time $T_{н}$

essence of the proposed algorithm is to determine the displacement of the position of dynamic equilibrium by identifying the nonlinear components of the movement of the device. Nonlinearity is manifested the more, the greater the amplitude of oscillation of the moving part of the device. Consequently, with an increase in the initial deviation of the OS from the center of oscillations, the error in determining the vertical by the proposed method decreases. However, this reduction has a limit, due to the fact that the analytical law of motion of the OS, which forms the basis of the algorithm for determining the vertical, is obtained by solving a nonlinear equation in which the trigonometric function $\sin(\alpha)$ is

replaced by the first two terms of its expansion in a Taylor series. Therefore, strictly speaking, the analytical law of motion of the form (5) is incorrect for large values of the amplitudes of the oscillations and, therefore, the algorithm built on its basis, with large amplitudes, gives a large error that increases with the amplitude. Graphically, the dependence of the error $\Delta\hat{\alpha}_0$ from the initial angles of deviation of the CS with respect to the center of oscillations at certain fixed disturbing moments is shown in fig. 4.
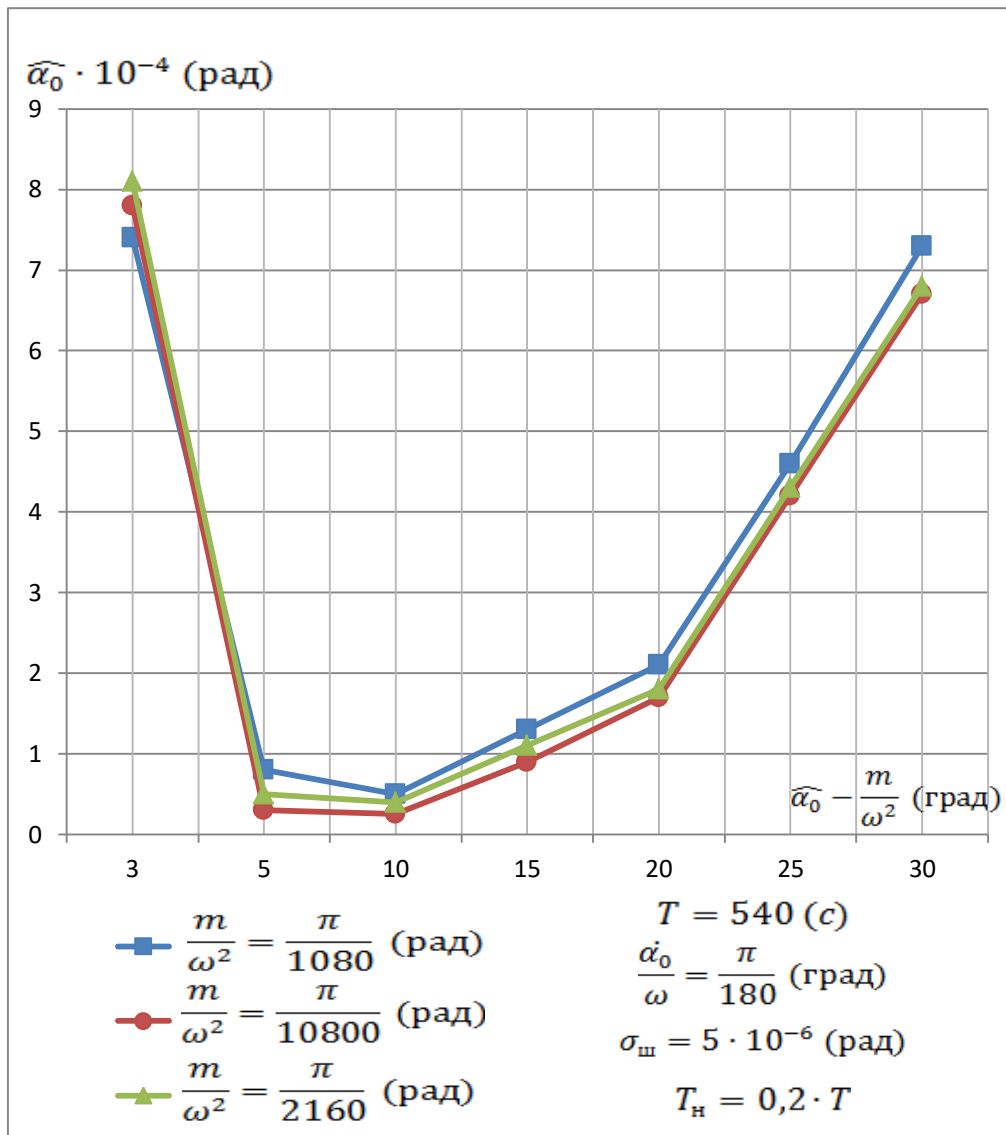

Fig. 4. Dependence of error $\Delta\hat{\alpha}_0$ estimates $\hat{\alpha}_0$ from the angle of deviation of the OS with respect to the equilibrium position of the oscillation

The effectiveness of the proposed method

The analysis also shows that the proposed method is not effective for all vibrations. If we take for comparison as a basic variant of the representation of

the motion of a CS in the form of an unbiased harmonic, then the error in determining the meridian by the basic method is m/ω^2. In order to determine the effectiveness of the proposed method in comparison with the baseline, the dependence of the value $\Delta\hat{\alpha}_0/m/\omega^2$ from the magnitudes of perturbing moments and angles of deviation of the OS from the center of oscillation. Where the reduced ratio is less than one ($\Delta\hat{\alpha}_0 < \dfrac{m}{\omega^2}$), the use of the proposed method is expedient, with the other values of amplitudes and perturbations the basic option is more preferable. From fig. 5 it follows that the efficiency of the method proposed in the work increases with the increase of the constant perturbation moment, as well as with the approximation of the initial deviation of the COP from the center of oscillation to 10 °.
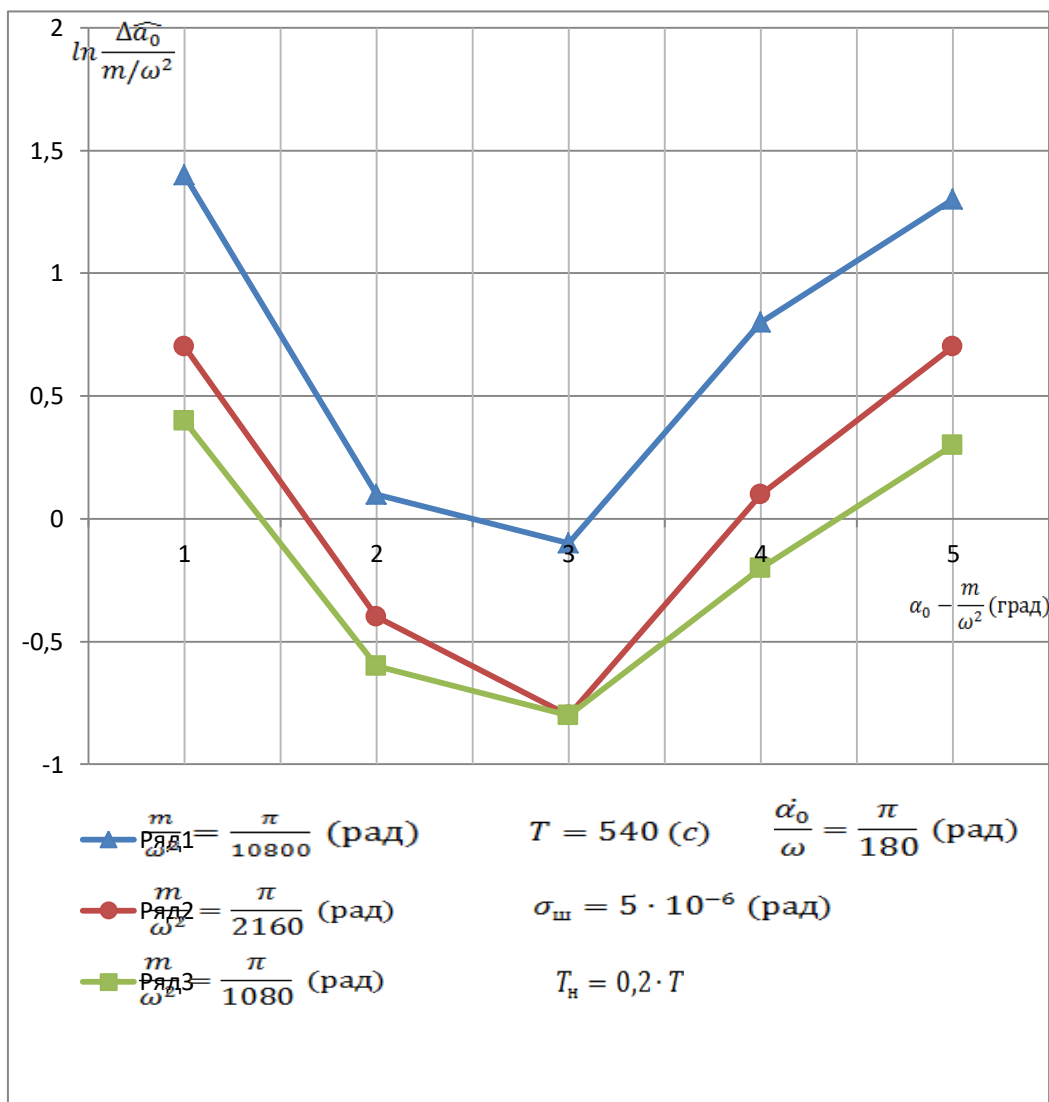

Fig. 5. The dependence of the magnitude $\Delta\hat{\alpha}_0/m/\omega^2$ from the magnitudes of the disturbing moments and angles of deviation of the OS with respect to the equilibrium position of the oscillations

Ways to improve the proposed methodology

As follows from the foregoing, the proposed method is fundamentally capable of determining in one measurement both the initial deviation of the pendulum from the vertical and the magnitude of the uncontrolled moment. The low accuracy of this definition is due to the fact that the algorithm for estimating the initial deflection of the pendulum is built on an insufficiently accurate model of the motion of the pendulum (only the first two terms of the expansion in the series are retained in the equation of motion). If, as a mathematical model of the motion of a OS, we consider not the analytical dependence (5), but the result of integrating the differential equation (1), then the above complexity can be circumvented [5]. In this case, the computational tools integrate equation (1) for all possible values of the initial conditions and the magnitudes of the uncontrollable moment and will make a comparison using the least squares method with the actual motion of the pendulum.
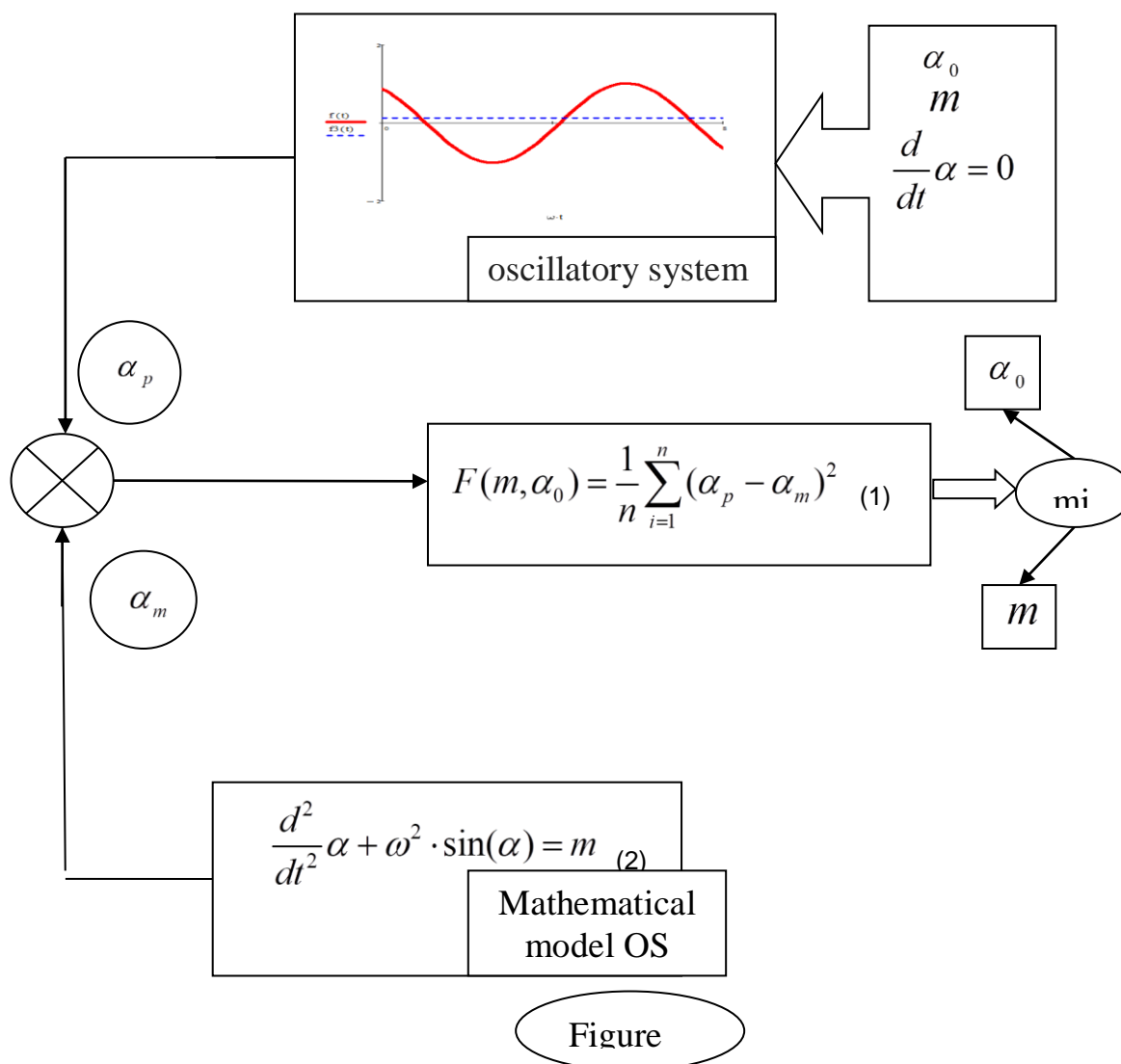
Fig. 6. Scheme of the machine experiment

Findings

1. In work the pendulum oscillatory system (OS) is considered, around which axis of rotation a constant uncontrollable moment acts. It is shown that, analyzing the nonlinear properties of the CS, it is possible to determine both the indicated time and the initial deviation of the CS during one measurement.

2. Based on the least squares method, an information processing algorithm has been developed that takes into account the first two terms of the harmonic expansion in a row. Explanation of the limitations inherent in the developed algorithm and the reasons causing these limitations are explained.

3. It is indicated that the further development of this method can be the use of OLS in its machine form, which will make it possible to analyze the nonlinear properties of a CS in full volume.

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