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BENDING AND NATURAL VIBRATION OF THE TRIPLE LAYER PLATES WITH ELASTIC FOUNDATION

Introduction

Using of the composite plates in loaded structures is one of the ways to improve the weight characteristics of spacecraft. The layered structural elements are widely used in transport engineering and construction practice. Three-layer plates are the most efficient in terms of strength and rigidity under bending deformation.

A much attention is given to the development of the theory and methods of calculation the layered structural elements of strength and stability at various influences. The complexity of research the stressed-strained state of layered systems stimulates the development of applied two-dimensional theories. Among these theories the principal place is occupied by the theory which based on the using the hypotheses for the whole package of layers in general.

Thus, the calculation of sandwich structures under static loads is an actual task.

Formulation and solution of the problem

Elastic three-layer plate with a rigid filler is resilient manner. To describe the kinematics package we adopted the hypothesis of broken line: the bearing layers are valid Kirchhoff's hypothesis; in incompressible filler thickness remains normal straight and does not change its length, but returns some extra angle that makes with the coordinate axes value $\psi_x(x,y), \psi_y(x,y)$ [1, 2]. On the plate are distributed external surface load $q(x,y)$, $p_x(x,y)$, $p_y(x,y)$ and the reaction base. Reaction basis corresponds to the model of Winkler [3].

$$q_r = -kw \qquad (1)$$

where k is the stiffness of base; w is the plate deflection, a minus sign indicates that the response is directed in the direction opposite deflection.

Using the introduced geometric hypotheses, the longitudinal displacement in layer can be expressed by five unknown functions $u_x, u_y, \psi_x, \psi_y, w$. [1]

$$u_x^{(1)} = u_x + c\psi_x - zw,_x, \quad u_y^{(1)} = u_y + c\psi_y - zw,_y, \quad (c \le z \le c + h_1),$$

$$u_x^{(3)} = u_x + z\psi_x - zw,_x, \quad u_y^{(3)} = u_y + z\psi_y - zw,_y, \quad (-c \le z \le c),$$

$$u_x^{(2)} = u_x - c\psi_x - zw,_x, \; u_y^{(1)} = u_y - c\psi_y - zw,_y, (-c - h_2 \le z \le -c) \qquad (2)$$

where z is the distance from the considered fibers to the median plane of the filler; $u + c\psi$ is the deflection of external base layer, the second base layer the deflection will be correspondingly $(u - c\psi)$

The components of strain tensor expressed in five unknown functions using Cauchy relations and expressions (2):

$$\varepsilon_{xx}^{(1)} = u_x,_x + c\psi_x,_x - zw,_{xx}, \quad \varepsilon_{yy}^{(1)} = u_y,_y + c\psi_y,_y - zw,_{yy}, \quad (c \le z \le c + h_1)$$

$$\varepsilon_{xx}^{(3)} = u_x,_x + z\psi_x,_x - zw,_{xx}, \quad \varepsilon_{yy}^{(3)} = u_y,_y + z\psi_y,_y - zw,_{yy}, \quad (-c \le z \le c)$$

$$\varepsilon_{xx}^{(2)} = u_x,_x - c\psi_x,_x - zw,_{xx}, \quad \varepsilon_{yy}^{(2)} = u_y,_y - c\psi_y,_y - zw,_{yy}, \; (-c - h_2 \le z \le -c)$$

$$\varepsilon_{xz}^{(1)} = \varepsilon_{xz}^{(2)} = 0, \; \varepsilon_{xz}^{(3)} = \frac{1}{2}\psi_x, \quad \varepsilon_{yz}^{(1)} = \varepsilon_{yz}^{(2)} = 0, \quad \varepsilon_{yz}^{(3)} = \frac{1}{2}\psi_y, \quad \varepsilon_{yx}^{(k)} = \varepsilon_{xy}^{(k)}. \qquad (3)$$

Spherical and deviator components of strain tensor in this case will be the following

$$(ý_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij}; \quad i, j = x, y, z):$$

$$\varepsilon^{(k)} = \frac{1}{3}(\varepsilon_{xx}^{(k)} + \varepsilon_{yy}^{(k)}), \quad ý_{xx}^{(k)} = \frac{2}{3}\varepsilon_{xx}^{(k)} - \frac{1}{3}\varepsilon_{yy}^{(k)}, \quad ý_{yy}^{(k)} = \frac{2}{3}\varepsilon_{yy}^{(k)} - \frac{1}{3}\varepsilon_{xx}^{(k)},$$

$$ý_{xz}^{(3)} = \varepsilon_{xz}^{(3)}, \quad ý_{yz}^{(3)} = \varepsilon_{yz}^{(3)}, \quad ý_{xy}^{(k)} = \varepsilon_{xy}^{(k)}, \qquad (4)$$

We introduce the internal forces and moments by following relationship:

$$N_{xx}^{(k)} = \sum_{k=1}^{3} \int_{h_k} \sigma_{xx}^{(k)} dz, \; N_{yy}^{(k)} = \sum_{k=1}^{3} \int_{h_k} \sigma_{yy}^{(k)} dz,$$

$$Q_x = \int_{h_3} \sigma_{xz}^{(3)} dz, \; Q_y = \int_{h_3} \sigma_{yz}^{(3)} dz, \; Q_{xy}^{(k)} = \sum_{k=1}^{3} \int_{h_k} \sigma_{xy}^{(k)} dz, \qquad (5)$$

$$M_{xx}^{(k)}=\sum_{k=1}^{3}\int_{h_k}\sigma_{xx}^{(k)}zdz,\ M_{yy}^{(k)}=\sum_{k=1}^{3}\int_{h_k}\sigma_{yy}^{(k)}zdz,\ M_{xy}^{(k)}=\sum_{k=1}^{3}\int_{h_k}\sigma_{xy}^{(k)}zdz,$$

where $\sigma_{xx}^{(k)},\sigma_x,\sigma_{yy}^{(k)},\sigma_y$ are the components of the stress tensor in the layers of the plate; integrals taken along the thickness of the k layer.

Balance equation derived from a variation principle of Lagrange:

$$\delta A+\delta W=0. \tag{6}$$

This variation of the external surface forces:

$$\delta A=\iint_S\left(p\delta u_x+p\delta u_y+\left(q+q_r\right)\delta w\right)dS \tag{7}$$

Variation of internal stress take into account work of filler in a tanhen direction:

$$\delta W=\iint_S\left\{\left[\sum_{k=1}^{3}\int_{h_k}\left(\sigma_{xx}^{(k)}\delta\varepsilon_{xx}^{(k)}+\sigma_{yy}^{(k)}\delta\varepsilon_{yy}^{(k)}+2\sigma_{xy}^{(k)}\delta\varepsilon_{xy}^{(k)}\right)dz\right]+\right.$$
$$\left.+2\int_{h_3}\left(\sigma_{xx}^{(k)}\delta\varepsilon_{xx}^{(k)}+\sigma_{yy}^{(k)}\delta\varepsilon_{yy}^{(k)}\right)dz\right\}dxd. \tag{8}$$

Variations of displacements in layers are:

$$\delta u_x^{(1)}=\delta u_x+c\delta\psi_x-z\delta w,_x,$$
$$\delta u_y^{(1)}=\delta u_y+c\delta\psi_y-z\delta w,_y\quad (c\le z\le c+h_1),$$
$$\delta u_x^{(3)}=\delta u_x+z\delta\psi_x-z\delta w,_x,$$
$$\delta u_y^{(3)}=\delta u_y+z\delta\psi_y-z\delta w,_y\quad (-c\le z\le c),$$
$$\delta u_x^{(2)}=\delta u_x-c\delta\psi_x-z\delta w,_x,$$
$$\delta u_y^{(1)}=\delta u_y-c\delta\psi_y-z\delta w,_y\quad (-c-h_2\le z\le -c). \tag{9}$$

we obtain the system differential equations of equilibrium three-layered rectangular plates on elastic foundation in the effort:

$$N_{xx},_x+Q_{xy},_y=-p_x,\quad N_{yy},_y+Q_{xy},_x=-p_y,\quad H_{xx},_x+H_{xy},_y-Q_x=0 \tag{10}$$
$$H_{yy},_y+H_{xy},_x-Q_y=0,\quad M_{xx},_{xx}+2H_{xy},_{xy}+M_{yy},_{yy}=-(q+q_r).$$

To contact stresses and strains in the layers we using the ratio of Hooke's law:

$$S_{ij}^{(k)}=2G_k\acute{y}_{ij}^{(k)},\quad \sigma^{(k)}=3K_k\varepsilon^{(k)},\quad (k=1,2,3;i,j=x,y,z), \tag{11}$$

where G_k, K_k are the module sliding and volumetric strain; $S_{ij}^{(k)}, \sigma^{(k)}$ are the deviator and ball components of stress tensor; $ý_{ij}^{(k)}, \varepsilon^{(k)}$ are the components of strain tensor.

The components of the stress tensor in layers considering the expressions (5) and (11) will be:

$$\sigma_{xx}^{(k)} = S_x^{(k)} + \sigma^{(k)} = \frac{4}{3}G\varepsilon_{xx}^{(k)} + K\varepsilon_{xx}^{(k)} - \frac{2}{3}G\varepsilon_{yy}^{(k)} + K\varepsilon_{yy}^{(k)} = K_k^+\varepsilon_{xx}^{(k)} + K_k^-\varepsilon_{yy}^{(k)};$$

$$\sigma_{yy}^{(k)} = K_k^-\varepsilon_{xx}^{(k)} + K_k^+\varepsilon_{yy}^{(k)}; \qquad \sigma_{xy}^{(k)} = 2G_k\varepsilon_{xy}^{(k)}; \qquad (12)$$

$$\sigma_{xz}^{(3)} = 2G_3\varepsilon_{xz}; \qquad \sigma_{yz}^{(3)} = 2G_3\varepsilon_{yz};$$

where $K_k^+ = K_k + \frac{4}{3}G_k;\ K_k^- = K_k - \frac{2}{3}G_k.$

Substituting in (8) expression strain through the desired movement (3), and using integration over the thickness of each layer, and taking into account expressions (7), (6) and (1) we obtain from (10) the system of five linear differential equations of equilibrium on the unknown displacements:

$$\begin{aligned}
&a_1\left(u_x,_{xx} + u_y,_{yx}\right) + a_2\left(\psi_x,_{xx} + \psi_y,_{yx}\right) - a_3\left(w,_{xxx} + w,_{yyx}\right) + a_8 u_x,_{yy} + \\
&\quad + a_9\psi_x,_{yy} = -p_x, \\
&a_1\left(u_y,_{yy} + u_x,_{xy}\right) + a_2\left(\psi_y,_{yy} + \psi_x,_{xy}\right) - a_3\left(w,_{yyy} + w,_{xxy}\right) + a_8 u_y,_{xx} + \\
&\quad + a_9\psi_y,_{xx} = -p_y, \\
&a_2\left(u_x,_{xx} + u_y,_{yx}\right) + a_4\left(\psi_x,_{xx} + \psi_y,_{yx}\right) - a_5\left(w,_{xxx} + w,_{yyx}\right) + a_9 u_x,_{yy} + \\
&\quad + a_{10}\psi_x,_{yy} - a_7\psi_x = 0, \\
&a_2\left(u_y,_{yy} + u_x,_{xy}\right) + a_4\left(\psi_y,_{yy} + \psi_x,_{xy}\right) - a_5\left(w,_{yyy} + w,_{xxy}\right) + a_9 u_y,_{xx} + \\
&\quad + a_{10}\psi_y,_{xx} - a_7\psi_y = 0, \\
&a_3\left(u_x,_{xxx} + u_y,_{yxx} + u_x,_{xyy} + u_y,_{yyy}\right) + a_5\left(\psi_x,_{xxx} + \psi_y,_{yxx} + \psi_x,_{xyy} + \psi_y,_{yyy}\right) - \\
&\quad - a_6\left(w,_{xxxx} + w,_{yyyy} + 2w,_{yyxx}\right) + kw = -q,
\end{aligned} \qquad (13)$$

where

$$a_1 = \sum_{k=1}^{3} h_k K_k^+,$$

$$a_2 = c(h_1K_1^+ - h_2K_2^+);$$

$$a_3 = h_1\left(c + \frac{h_1}{2}\right)K_1^+ - h_2\left(c + \frac{h_2}{2}\right)K_2^+;$$

$$a_4 = c^2\left(h_1K_1^+ + h_2K_2^+\right) + \frac{2}{3}c^3K_3^+;$$

$$a_5 = c\left[h_1\left(c + \frac{h_1}{2}\right)K_1^+ + h_2\left(c + \frac{h_2}{2}\right)K_2^+ + \frac{2}{3}c^3K_3^+\right];$$

$$a_6 = h_1\left[c^2 + ch_1 + \frac{h_1^2}{3}\right]K_1^+ + h_2\left[c^2 + ch_2 + \frac{h_2^2}{3}\right]K_2^+ + \frac{2}{3}c^3K_3^+;$$

$$a_7 = 2G_3c;$$

$$a_8 = \sum_{k=1}^{3} h_kG_k;$$

$$a_9 = G_1c(h_1 - h_2);$$

$$a_{10} = c^2(G_1h_1 - G_2h_2) + \frac{2}{3}c^3G_3.$$

In this paperwe using a numerical approach to solving the problem.

Considering a three-layer construction, which is the middle surface of $\Omega \subset R2$. The problem of determining the stress-strain state in the variational formulation based on the minimum potential energy can be formulated as a problem of minimizing a quadratic functional:

$$u \in V, \quad \acute{Y}(u) = \inf \acute{Y}(v); \quad v \in V, \tag{14}$$

where V is the space of admissible displacements. Elements belonging to the space V, satisfy the kinematic boundary conditions of the problem and the requirements of the smoothness of the desired solution. Function $\acute{Y}(v)$ represents a potential energy system

$$\acute{Y}(v) = \frac{1}{2}\alpha(v,v) - f(v), \tag{15}$$

where $\alpha(v,v)$are a symmetric bilinear form. From the energy point of view, $\alpha(v,v)$ determines the potential energy of the elastic deformation of the structure, $f(v)$ is the work of external forces.

In the calculation of the stress-strain state of structural components ,exposed to vibration loads, it’s necessary to determine the natural frequencies and corresponding mode shapes.The finding of the fundamental natural frequency can be reduced to a minimization problem, where the functional is determined by the ratio of the Rayleigh-Ritz method

$$\omega^2 = \min\frac{\Pi(v)}{\mathrm{T}(v)}, \quad v \in V. \tag{16}$$

Here $\Pi(v)$ is the amplitude of the strain energy, $\mathrm{T}(v)$ is a quantity proportional with the factor ω^2 to peak value of kinetic energy. In the construction of variational-difference schemes for the three-layer sandwich structures we using triangular element, which, in contrast to previously developed models [4], is used approximation movements on different layers. To approximate the deflection of thin bearing is used incomplete cubic polynomial. Mid-plane displacement of the points of bearing layers along the x and y are defined as linear polynomials within each triangle. Moving points of the middle plane of the filler u_3, v_3, w_3 easily expressed in terms of the displacements of the middle surfaces bearing layers, based on the conditions of sandwich construction work without slip between the layers.

Applying adopted approximation of displacements, the functional (15), (16) on the finite-dimensional space V_h admissible functions are as follows

$$\acute{Y}\left(v_h\right)=\frac{1}{2}\alpha\left(v_h,v_h\right)-f\left(v_h\right), \quad v_h\in V_h. \tag{17}$$

$$F\left(v_h\right)=\frac{\Pi\left(v_h\right)}{\mathrm{T}\left(v_h\right)}; \quad v_h\in V. \tag{18}$$

To minimize the functional (17), (18) we proposed to use the method of coordinate descent [5]. The choice of this method is due to the fact that its using is needn't in the formation and storage of mass and stiffness matrices of large dimensions, the numbering of nodes in a sampling area is arbitrary.

Conclusion

We adduced basic equations for the problem bending vibrations of a three-centered plate with on elastic foundation. Considering the stiffness of elastic foundation leads to a significant refinement of the stress state that occurs in a three-layer plate with external force action. We consider a numerical method for determining the stress-strain state and the natural vibrations of sandwich plates on elastic foundation. The method of coordinate descent is an iterative method and stable rounding errors have little effect on the accuracy of the final result. When solving a series of practical problems of bending and vibration the maximum number of iterations was not more than 140. The error in determining the deflection at the center of the plates is 0.2%. Data to determine the maximum deflection and fundamental natural frequencies are compared with the analytical results obtained in [1-3]. After 120 iterations, the maximum error in the determination of basic natural frequency of 4.3%.

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